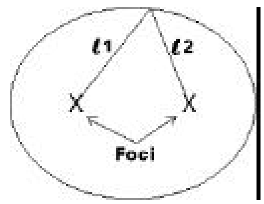
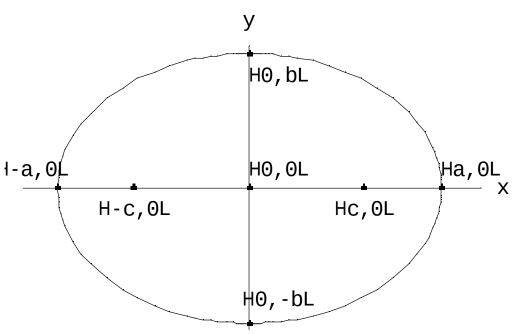
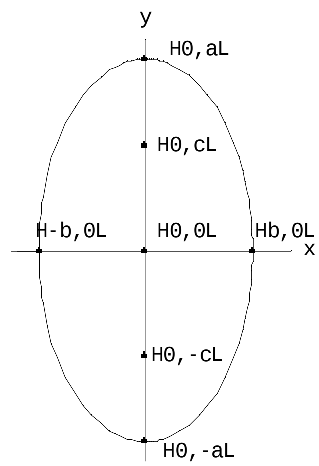
A1 Ellipses

An *ellipse* is the set of all points (x, y) in the plane such that the sum of the distances l_1 and l_2 from two fixed points, called *foci* (plural for *focus*), is constant.



There are two standard equations for the ellipse depending on whether it is elongated horizontally or vertically. Note that in the equations below it is assumed that $0 < b \leq a$.





$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

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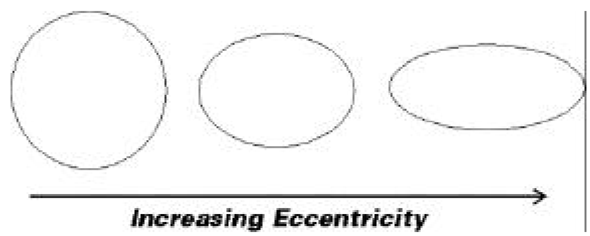
The line that passes through the foci, $(-c,0)$ and $(c,0)$ (or $(0,-c)$ and $(0,c)$, respectively) intersects the ellipse at its *vertices*, $(-a,0)$ and $(a,0)$ (or $(0,-a)$ and $(0,a)$, respectively). The point halfway between the vertices (and the foci) is the *center* (here, the center is the origin). The chord with length $2a$ joining the vertices is called the *major axis*. The chord perpendicular to the major axis through the center is called the *minor axis* and has length $2b$. (When $a = b$, we have a circle, which is a special case.) Notice that an ellipse is symmetrical about its major and minor axes.

In the first standard equation, notice that a^2 is the denominator of the fraction containing x^2 as the numerator. The ellipse on the left is elongated horizontally, i.e., along the x-axis. Similarly, the ellipse on the right is elongated vertically along the y-axis; thus, a^2 is the denominator of the fraction containing y^2. The values of a, b, and c are related such that $c^2 = a^2 - b^2$.

There is a simple method for constructing the graph of an ellipse. First, place a sheet of paper over a sheet of cardboard and insert two thumbtacks into the paper and board. The tacks will be the foci of the ellipse. Now, attach the ends of a piece of string that is longer than the distance between the foci to each of the thumbtacks. With a pencil, trace the path of the ellipse by pushing the pencil against the string, keeping the string taut at all times. Notice that l_1 gets longer by the same amount that l_2 gets shorter since the length of the string is always $l_1 + l_2$, a constant, according to the definition of an ellipse.

If the foci are relatively close together and the string is several times the length of the segment joining the foci ($2a$ is much greater than $2c$), then the ellipse will look very close to a circle. Inversely, if the foci are relatively far apart and the string is only slightly longer than the segment joining the foci, i.e., c and a are approximately the same, then the ellipse will be elongated or look "squished."



Eccentricity as a numerical measure of how much the ellipse differs from a circle is defined as $e = \frac{c}{a}$ with e = 1 for a circle.

Ellipses have a fascinating characteristic associated with their foci. If an object is launched from one focus and collides with the ellipse, it will be reflected in such a way that it will move toward the other focus! This idea was used in designing "whispering galleries." The walls and ceilings of these rooms were all made elliptical. As a result, a person standing at one focus will be able to hear the whispers of a person standing at the other focus. In this case, the reflected objects are the sound waves. Some of the most famous whispering galleries are located in St. Paul's Cathedral in London, the Capitol in Washington D.C., and the Mormon Tabernacle in Salt Lake City.[1]

[1] David B. Johnson and Thomas A. Mowry, *Mathematics: A Practical Odyssey*, 1995.