Introduction to Modeling

By the term "modeling", we shall mean the translation of a so-called "word problem" into a concrete mathematical problem. In particular, we will look at problems in which some quantity must be **optimized** (either maximized or minimized). Hence, the concrete mathematical problem that we reduce to will always be one of finding a **global extremum** of some function on some domain. In this short note, I want to outline a strategy for modeling in such settings. Specifically, we will develop a method for following $\longrightarrow$ below.

$$\left[\begin{array}{c}\text{Solve an optimization}\\ \text{word problem}\end{array}\right] \longrightarrow \left[\begin{array}{c}\text{Find a global extremum of a}\\ \text{function } f \text{ on an interval } I\end{array}\right]$$

Here is a six step outline of our strategy, after which we will see how well this works with a few examples.

0 Draw a picture.

1 What are the variables? Give them names ($x, y, z, ...$ say). Label them on the picture.

2 What quantity is being optimized? Give it a name (Q say). Label it on the picture.

3 Express Q in terms of some or all of the variables $x, y, z, ...$ and find **constraints** among these variables.

4 Use the constraints in 3 to eliminate **all but one** variable (x say) and hence express Q as a function **of x alone**.

5 Find the appropriate global extremum of the function $Q(x)$ on a suitable interval.

Example 1. What is the maximum product of two positive numbers whose sum is 100?

0 Skip: we don't need a picture here.

1 The variables are the two numbers. Call them x and y.

2 We're trying to **maximize the product** of the two numbers. Call the product P.

3 P is expressed in terms of x and y as follows: $P = xy$. The **constraint** between x and y is that their sum must be 100. That is $x + y = 100$.

4 Let's eliminate y. Using the constraint, we get $y = 100 - x$. Now substituting into our expression for P we get $P = x(100 - x)$, so that $P(x) = 100x - x^2$ is a function **of x alone**.

5 A suitable domain for $P(x)$ is the set of x-values for which the problem makes sense. Since both x and y must be positive, x must be in the closed interval $[0, 100]$. Hence, the original word problem is equivalent to the problem of finding the **global maximum** of the function $P(x)$ on the closed interval $[0, 100]$.

Now finish the job off yourself: does the answer that you get agree with your intuition?

Example 2. What is the minimum sum of two positive numbers whose product is 45?

<u>0</u> Skip: we don't need a picture here.

<u>1</u> The variables are the two numbers. Call them x and y.

<u>2</u> We're trying to **minimize the sum** of the two numbers. Call the sum S.

<u>3</u> S is expressed in terms of x and y as follows: $S = x + y$. The **constraint** between x and y is that their product must be 45. That is $xy = 45$.

<u>4</u> Let's eliminate y. Using the constraint, we get $y = 45/x$. Now substituting into our expression for S we get $S = x + 45/x$, a function **of x alone**.

<u>5</u> Here, if x is very small, then y would be very large (since their product is 45) and vice versa. We might as well just look at all positive numbers. The original word problem is equivalent to the problem of finding the **global minimum** of the function $S(x) = x + 45/x$ on the open interval $(0, \infty)$.

Again, finish the job off yourself: did you anticipate the answer that you got?

Try the next one yourself:

Example 3. A farmer wishes to enclose a rectangular region in which to pen his sheep. He has 200ft of fencing. What dimensions should he choose in order to maximize the area of the region?

Notes:

- You should include a picture this time (step <u>0</u>).
- The optimization problem that you get at step <u>5</u> should look very familiar. Does it?

Concluding Remark: Once you're at step <u>5</u>, the methods you learnt in 4.3 will carry you through. Recall that in 4.3, we considered two important settings in which to find global extrema of continuous functions: on a **closed** interval; and on an **open** interval or the entire **real line**. Notice that step <u>5</u> in Example 1 was of the first type, while step <u>5</u> in Example 2 was of the second.