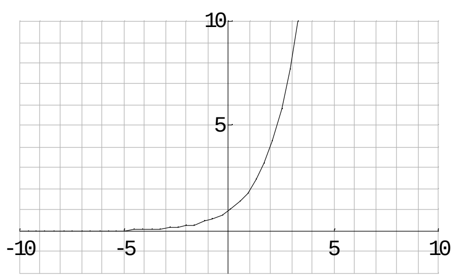
3.4 Exponential Functions

Exponential functions are of the form

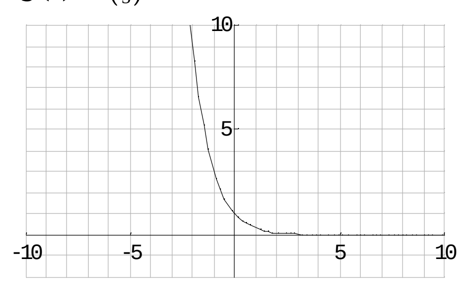
$$f(x) = a \cdot b^x + c,$$

where a, b, and c are constants and $b > 0$. Graphs of exponential functions are either always increasing or always decreasing. Furthermore, they level off as the input values become very large or very small. At the opposite end of the graph, output values increase or decrease without bound. There is no change of curvature. Here are some graphs of exponential functions. (The number e in the functional expression of part c) is called Euler's number and has an approximate value of 2.71828.)

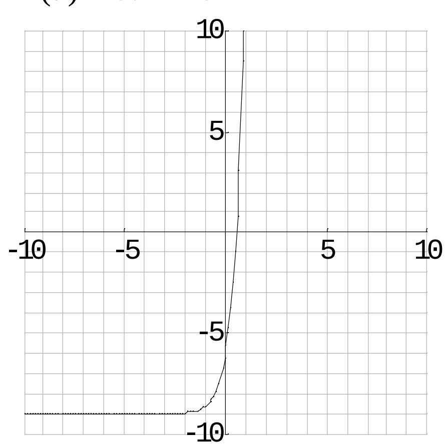
a) $f(x) = 2^x$



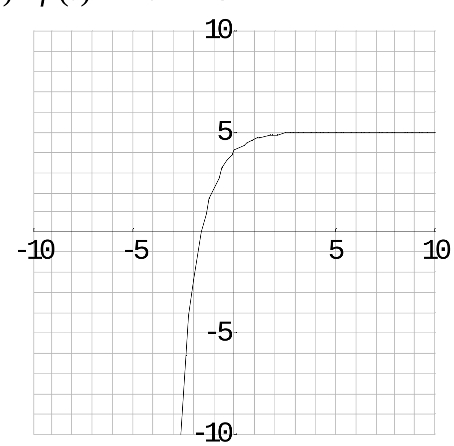
b) $g(t) = \left(\frac{1}{3}\right)^t$



c) $h(u) = 3e^{2u} - 9$



d) $f(t) = -e^{-t} + 5$



We start by identifying the values of the parameters a, b, and c for these functions. Furthermore, you will make a conjecture on how the value of the parameter c influences the shape of the graph.

Activity 3.4.1

For the functions above determine the values of the parameters a, b, and c. Can you see what influence the value of c has on the graph?

a) $f(x) = 2^x$ $a =$ $b =$ $c =$

b) $g(t) = \left(\frac{1}{3}\right)^t$ $a =$ $b =$ $c =$

c) $h(u) = 3e^{2u} - 9 = 3(e^2)^u - 9$ $a =$ $b =$ $c =$

d) $f(t) = -e^{-t} + 5$ $a =$ $b =$ $c =$

Influence of c:

Exponential functions of the form

$$f(x) = a \cdot b^x,$$

where $c = 0$, have the special feature that output values increase or decrease in equal proportions in equal time. This means there is a constant factor of growth or decay. Here is an example:

x	$f(x) = 3 \cdot 2^x$
-2	$3 \cdot 2^{-2} = 3 \cdot \frac{1}{2^2} = \frac{3}{4}$
-1	$3 \cdot 2^{-1} = 3 \cdot \frac{1}{2^1} = \frac{3}{2}$
0	$3 \cdot 2^0 = 3 \cdot 1 = 3$
1	$3 \cdot 2^1 = 3 \cdot 2 = 6$
2	$3 \cdot 2^2 = 3 \cdot 4 = 12$
3	$3 \cdot 2^3 = 3 \cdot 8 = 24$

If we compare consecutive output values, then we see that they double for each unit increase in input values. This is somewhat similar to linear functions. For linear functions, a fixed amount is <u>added</u> per unit increase in input value. For exponential functions, a fixed multiplicative factor is applied, i.e., the output is <u>multiplied</u> by a fixed amount for each unit increase in input value. We can find this fixed multiplicative factor by looking at the ratios of output values. Thus, exponential functions of the form $a \cdot b^x$ are said to have *constant unit ratios*.

Computation of Unit Ratios (for exponential functions of the form $a \cdot b^x$):

Select consecutive input values with $x_1 \le x_2$ and their respective output values $f(x_1)$ and $f(x_2)$. The unit ratio is given by

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{\frac{1}{\Delta x}}$$

where $\Delta x = x_2 - x_1$.

Remarks: 1) If the input values differ by only one unit, then this quotient reduces to $\dfrac{f(x_2)}{f(x_1)}$.

2) Note that the order of the input values is important here; the output value in the numerator belongs to the larger input value.

Example: $f(x) = 3 \cdot \left(\frac{1}{2}\right)^x$

Select increasing input values and compute the corresponding output values. (You may need to review rules for exponents in Appendix A3.) Next, compute the differences Δx for each pair of input values. This difference is obtained by subtracting the smaller value from the larger one. To calculate the unit ratios, compute the ratio of output values, then raise to the power $\frac{1}{\Delta x}$. Here is a worked out example for the points $(x_1, f(x_1)) = (-3, 24)$ and $(x_2, f(x_2)) = (-2, 12)$.

$$\Delta x = x_2 - x_1 = (-2) - (-3) = -2 + 3 = 1$$

$$\text{unit ratio} = \left(\frac{f(x_2)}{f(x_1)}\right)^{\frac{1}{\Delta x}} = \left(\frac{12}{24}\right)^{\frac{1}{1}} = \left(\frac{1}{2}\right)^{1} = \frac{1}{2}.$$

Here is another example for input values $x_1 = 3$ and $x_2 = 6$.

$$f(x_1) = 3 \cdot \left(\frac{1}{2}\right)^3 = 3 \cdot \frac{1}{8} = \frac{3}{8}$$

$$f(x_2) = 3 \cdot \left(\frac{1}{2}\right)^6 = 3 \cdot \frac{1}{64} = \frac{3}{64}$$

$$\Delta x = x_2 - x_1 = 6 - 3 = 3 \quad \Rightarrow \quad \frac{1}{\Delta x} = \frac{1}{3}$$

$$\text{unit ratio} = \left(\frac{f(x_2)}{f(x_1)}\right)^{\frac{1}{\Delta x}} = \left(\frac{3/64}{3/8}\right)^{\frac{1}{31}} = \left(\frac{3}{64} \cdot \frac{8}{3}\right)^{\frac{1}{3}} = \left(\frac{1}{8}\right)^{\frac{1}{3}} = \boxed{\frac{1}{2}}$$

x	$f(x)$	Δx	Unit Ratios $\left(\frac{f(x_2)}{f(x_1)}\right)^{1/\Delta x}$
-3	24		
		-2 - (-3) = 1	$\left(\frac{12}{24}\right)^{1/1} = \frac{1}{2}$
-2	12		
		$-1-(-2)=1$	$\left(\frac{6}{12}\right)^{1/1} = \frac{1}{2}$
-1	6		
		$0-(-1)=1$	$\left(\frac{3}{6}\right)^{1/1} = \frac{1}{2}$
0	3		
		$1-0=1$	$\left(\frac{3/2}{3}\right)^{1/1} = \left(\frac{3}{2}\cdot\frac{1}{3}\right) = \frac{1}{2}$
1	$\frac{3}{2}$		
		$3-1=2$	$\left(\frac{3/8}{3/2}\right)^{1/2} = \left(\frac{3}{8}\cdot\frac{2}{3}\right)^{1/2} = \left(\frac{1}{4}\right)^{1/2} = \frac{1}{2}$
3	$\frac{3}{8}$		
		6 - 3 = 3	$\left(\frac{3/64}{3/8}\right)^{1/3} = \left(\frac{3}{64}\cdot\frac{8}{3}\right)^{1/3} = \left(\frac{1}{8}\right)^{1/3} = \frac{1}{2}$
6	$\frac{3}{64}$		

Note that the unit ratio is equal to the value of b, in this case, $\frac{1}{2}$.

Activity 3.4.2

Compute the unit ratios for the function $f(x) = \frac{1}{2}\cdot 3^x$. Do not forget to raise the quotient to the power of $\frac{1}{\Delta x}$!

x	$f(x)$	Δx	Unit Ratios
-4			
-1			
0			
2			
3			

Let's summarize the properties of exponential functions:

Properties of Exponential Functions

1) The functional expression of an exponential function is given by $f(x) = a \cdot b^x + c$, with $b > 0$.
2) An exponential function is either always increasing or always decreasing, and there is no change in curvature.
3) At one end of the graph, the output values approach a horizontal line at level *c*. At the other end of the graph, the values either increase or decrease without bound.
4) Exponential functions of the form $f(x) = a \cdot b^x$ have constant unit ratios.

Using these properties, we can now develop a procedure for checking whether data follows an exponential function.

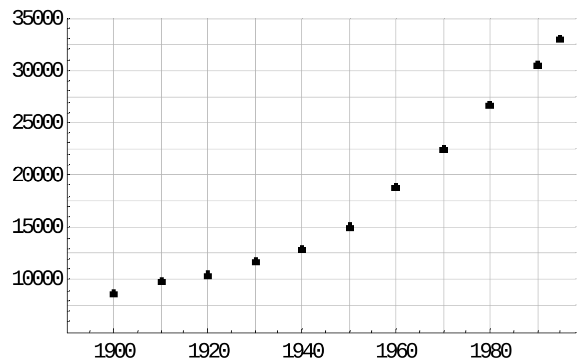
1. Graph the data.
2. Check whether the resulting graph is either increasing or decreasing, and shows no change of curvature.
3. Check whether the graph will level off at the horizontal axis. If not enough data is given to see this effect, think about the context of the data. If the data does not seem to level off at the horizontal axis, but at some other level $c \neq 0$, modify the data by first subtracting *c* from every output value.
4. Compute the unit ratios for the (modified) data. If the unit ratios are approximately constant, then the data is likely to come from an exponential function. You can use the average of the unit ratios as an estimate for the parameter *b*.

Remark: Very often, both an exponential or quadratic function seem plausible (if only one half of the quadratic function shows). In this case, it is difficult to decide between the two types from the shape of the graph only. However, once second unit differences and unit ratios are computed, one of the two types should become more likely. If both quantities are approximately constant, then we need additional statistical measures to decide on the function type (see Chapter 4).

Activity 3.4.3

The data[1] below gives the number of people in the US (in thousands) who are neither black nor white. From the graph, determine whether the data comes from an exponential function. Give reasons for your answer. If you think the data might be from an exponential function, determine the level of the horizontal asymptote, using the graph and the context of the data. (You do not need to compute the unit ratios!)

Year	Population
1900	8,834
1910	9,826
1920	10,463
1930	11,891
1940	12,866
1950	15,042
1960	18,872
1970	22,581
1980	26,683
1990	30,486
1995	33,095



[1] 1996 Statistical Abstract, Table No.12